

THE FARRELL-JONES ISOMORPHISM CONJECTURE FOR 3-MANIFOLD GROUPS

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ABSTRACT. We show that the Fibered Isomorphism Conjecture (FIC) of Farrell and Jones corresponding to the stable topological pseudoisotopy functor is true for the fundamental groups of a large class of 3-manifolds. We also prove that if the FIC is true for irreducible 3-manifold groups then it is true for all 3-manifold groups. In fact, this follows from a more general result we prove here, namely we show that if the FIC is true for each vertex group of a graph of groups with trivial edge groups then the FIC is true for the fundamental group of the graph of groups. This result is part of a program to prove FIC for the fundamental group of a graph of groups where all the vertex and edge groups satisfy FIC. A consequence of the first result gives a partial solution to a problem in the problem list of R. Kirby. We also deduce that the FIC is true for a class of virtually PD_3 -groups.

Another main aspect of this article is to prove the FIC for all Haken 3-manifold groups assuming that the FIC is true for B -groups. By definition a B -group contains a finite index subgroup isomorphic to the fundamental group of a compact irreducible 3-manifold with incompressible nonempty boundary so that each boundary component is of genus ≥ 2 . We also prove the FIC for a large class of B -groups and moreover, using a recent result of L.E. Jones we show that the surjective part of the FIC is true for any B -group.

CONTENTS

1. Introduction	2
2. Statement of the FIC and related results	3
3. Statement of Reduction Theorem: the FIC for free products and reduction to irreducible case	5

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4. Statements of theorems and proofs of basic results: the FIC for 3-manifold groups	7
5. Proofs of the reduction theorem and its' consequences	14
6. Proofs of Theorems 4.3, 4.4 and the Main Theorem	17
7. Proof of Theorem 4.5	23
8. Proofs of Theorems 4.6, 4.7, 4.8 and 4.9	24
9. Properties of the class \mathcal{C}	28
10. On A -regular Riemannian manifolds and 3-manifolds	30
11. Examples of 3-manifolds satisfying Condition*	32
References	34

1. INTRODUCTION

Algebraic K -theory of 3-manifold groups was considered by Waldhausen ([36]) leading to some very general results in algebraic K -theory which has far wide applications. Namely, the results of Waldhausen regarding algebraic K -theory of generalized free products and HNN -extensions. The motivation for these celebrated results was computing Whitehead groups of knots groups and of 3-manifold groups in general. This led him to prove the very important result that the Whitehead group, reduced projective class group and the negative K -groups all vanish for the fundamental group of any Haken 3-manifold. Another outstanding result in this direction is due to Farrell and Jones. They proved that the same result is true for the fundamental group of a hyperbolic manifold ([9], [10]) and more generally for nonpositively curved manifolds ([11]) (of any dimension). Thurston's Geometrization conjecture says that an irreducible aspherical 3-manifold is either Seifert fibered, hyperbolic or Haken. Plotnick ([29]) proved the vanishing of the above K -theoretic groups for non-Haken Seifert fibered spaces. Thus conjecturally this completes the picture for computing low dimensional algebraic K -theory of aspherical 3-manifold groups. In the setting of assembly map these K -theoretic vanishing results are implied by the isomorphism of some assembly map for pseudoisotopy functor. The more recent Fibered Isomorphism Conjecture (FIC) of Farrell and Jones predicts that this isomorphism to hold for all groups ([8]).

In this paper we are concerned about proving the FIC for several classes of 3-manifold groups and PD_3 groups. At first we establish a general result showing that the FIC is true for a free product if it is true for each free summand. From this result and some standard facts in 3-manifold topology we deduce that it is enough to consider irreducible

3-manifold groups only. Another aspect of this paper is to show that if the FIC is true for groups (we call such a group a B -group) containing a finite index subgroup isomorphic to the fundamental group of compact irreducible 3-manifold with incompressible nonempty boundary so that each boundary component is a surface of genus ≥ 2 , then it is true for all 3-manifold groups modulo Thurston's Geometrization conjecture. We also prove the FIC for a subclass of B -groups. We further show that any torsion free B -group can be obtained by applying generalized free products or HNN -extensions (amalgamated along infinite cyclic subgroup) on finitely many members of the above subclass.

In this paper by the FIC we always mean the Fibered Isomorphism Conjecture of Farrell and Jones ([8]) corresponding to the stable topological pseudoisotopy functor. In short it says that computing the pseudoisotopy functor of $K(\pi, 1)$ is equivalent to the computation of the pseudoisotopy functor of $K(V, 1)$ where V varies over all virtually cyclic subgroups of π . In recent times the FIC has been checked for several classes of groups and it has become an important fundamental tool for computing K -theory of groups (see for example [2], [8], [12], [13], [33], [34] etc.).

In the paper [32] F. Quinn mentioned that the technique we use to prove FIC in the pseudoisotopy functor case can also be used in the case of K -theory developed in [32] to deduce a similar conclusion for higher K -theory with arbitrary coefficient ring.

2. STATEMENT OF THE FIC AND RELATED RESULTS

In this section we recall the Fibered Isomorphism Conjecture of Farrell and Jones made in [8]. The following formulation of the conjecture is taken from [12].

Let \mathcal{S} denotes one of the three functors from the category of topological spaces to the category of spectra: (a) the stable topological pseudo-isotopy functor $\mathcal{P}()$; (b) the algebraic K -theory functor $\mathcal{K}()$; (c) and the L -theory functor $\mathcal{L}^{-\infty}()$.

Let \mathcal{M} be the category of continuous surjective maps. The objects of \mathcal{M} are continuous surjective maps $p : E \rightarrow B$ between topological spaces E and B . And a morphism between two maps $p : E_1 \rightarrow B_1$ and $q : E_2 \rightarrow B_2$ is a pair of continuous maps $f : E_1 \rightarrow E_2$, $g : B_1 \rightarrow B_2$ such that the following diagram commutes.

There is a functor defined by Quinn [31] from \mathcal{M} to the category of Ω -spectra which associates to the map $p : E \rightarrow B$ the spectrum $\mathbb{H}(B, \mathcal{S}(p))$ with the property that $\mathbb{H}(B, \mathcal{S}(p)) = \mathcal{S}(E)$ when B is a single point. For an explanation of $\mathbb{H}(B, \mathcal{S}(p))$ see [[8], section 1.4].

Also the map $\mathbb{H}(B, \mathcal{S}(p)) \rightarrow \mathcal{S}(E)$ induced by the morphism: $\text{id}: E \rightarrow E$; $B \rightarrow *$ in the category \mathcal{M} is called the Quinn assembly map.

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ \downarrow p & & \downarrow q \\ B_1 & \xrightarrow{g} & B_2 \end{array}$$

Let Γ be a discrete group and \mathcal{E} be a Γ space which is universal for the class of all virtually cyclic subgroups of Γ and denote \mathcal{E}/Γ by \mathcal{B} . For definition of universal space see [[8], appendix]. Let X be a space on which Γ acts freely and properly discontinuously and $p: X \times_{\Gamma} \mathcal{E} \rightarrow \mathcal{E}/\Gamma = \mathcal{B}$ be the map induced by the projection onto the second factor of $X \times \mathcal{E}$.

The Fibered Isomorphism Conjecture states that the map

$$\mathbb{H}(\mathcal{B}, \mathcal{S}(p)) \rightarrow \mathcal{S}(X \times_{\Gamma} \mathcal{E}) = \mathcal{S}(X/\Gamma)$$

is a (weak) equivalence of spectra. The equality is induced from the map $X \times_{\Gamma} \mathcal{E} \rightarrow X/\Gamma$ and using the fact that \mathcal{S} is homotopy invariant.

Let Y be a connected CW -complex and $\Gamma = \pi_1(Y)$. Let X be the universal cover \tilde{Y} of Y and the action of Γ on X is the action by group of covering transformation. If we take an aspherical CW -complex Y' with $\Gamma = \pi_1(Y')$ and X is the universal cover \tilde{Y}' of Y' then by [[8], corollary 2.2.1] if the FIC is true for the space \tilde{Y}' then it is true for \tilde{Y} also. Thus throughout the paper whenever we say that the FIC is true for a discrete group Γ or for the fundamental group $\pi_1(X)$ of a space X we would mean it is true for the Eilenberg-MacLane space $K(\Gamma, 1)$ or $K(\pi_1(X), 1)$ and the functor $\mathcal{S}()$.

Now we recall the relation between the Ω -spectra $\mathcal{P}()$ and lower algebraic K -theory. This result is proved in [1].

$$\pi_j(\mathcal{P}(M)) = \begin{cases} K_{j+2}(\mathbb{Z}\pi_1(M)) & \text{if } j \leq -3, \\ \tilde{K}_0(\mathbb{Z}\pi_1(M)) & \text{if } j = -2, \\ Wh(\mathbb{Z}\pi_1(M)) & \text{if } j = -1. \end{cases}$$

The following results are used frequently in this paper.

Recall that a wreath product $A \wr B$ of two groups A and B is by definition the semi-direct product $A^B \rtimes B$ with respect to the regular action of B on B . Also by the algebraic lemma of [13] if A is a normal subgroup of B with quotient C then B can be embedded in the wreath product $A \wr C$.

Definition 2.1. A group G is said to satisfy *FICwF* if the FIC is true for $G \wr K$ for any finite group K .

Lemma 2.1. ([8], theorem A.8]) *If the FICwF is true for a discrete group Γ then it is true for any subgroup of Γ .*

Lemma 2.2. ([8], proposition 2.2]) *Let $f : G \rightarrow H$ be a surjective homomorphism. Assume that the FIC is true for H and for $f^{-1}(C)$ for all virtually cyclic subgroup C of H (including $C = 1$). Then the FIC is true for G .*

Lemma 2.2 also holds if we replace FIC by FICwF.

Proposition 2.1. ([8], proposition 2.4]) *The FICwF is true for virtually poly- \mathbb{Z} groups.*

3. STATEMENT OF REDUCTION THEOREM: THE FIC FOR FREE PRODUCTS AND REDUCTION TO IRREDUCIBLE CASE

Let us first recall some basic definitions from 3-manifold topology. By a *3-manifold group* we mean the fundamental group of a 3-manifold. A 3-manifold is called *irreducible* if every embedding of the 2-sphere in the 3-manifold extends to an embedding of the 3-disc. And a 3-manifold is called *prime* if whenever it is a connected sum of two 3-manifolds then one of the summand is homeomorphic to the 3-spheres. The only 3-manifolds which are prime but not irreducible are the \mathbb{S}^2 bundles over the circle ([16], lemma 3.13). And by Kneser's prime decomposition theorem any compact 3-manifold can be written as a connected sum of finitely many prime 3-manifolds ([16], theorem 3.15). Thus we have the following lemma.

Lemma 3.1. *A compact 3-manifold group is a free product of finitely many irreducible 3-manifold groups and a free group of finite rank.*

The first theorem of this article is about the FIC for arbitrary groups.

Reduction Theorem. *Let G_1 and G_2 be two countable groups and assume that the FIC is true for both G_1 and G_2 . Then the FIC is true for the free product $G_1 * G_2$.*

Theorem 3.1. *Assume that the FICwF is true for G_i for $i = 1, 2$. Then the FICwF is true for $G_1 * G_2$.*

Corollary 3.1. *If the FICwF is true for compact irreducible aspherical 3-manifold groups then it is true for any 3-manifold group.*

Because of Corollary 3.1 and the Remark below in the next sections we always consider orientable irreducible aspherical 3-manifolds.

Remark 3.1. As any nonorientable 3-manifold is covered by orientable ones and we always prove the FICwF for 3-manifold groups, the nonorientable case follows using [[13], algebraic lemma].

A more interesting corollary to the Reduction Theorem is the following application to graph of groups. Recall that a *graph of groups* \mathcal{G} is a graph, that is a connected one dimensional *CW*-complex, with the following additional data.

- for each vertex v there is an associated group G_v .
- for each edge e there is an associated group G_e and for the end vertices v_i^e and v_f^e of e there are injective homomorphisms $G_e \rightarrow G_{v_i^e}$ and $G_e \rightarrow G_{v_f^e}$.

Note that v_i^e and v_f^e could be the same vertex.

In the above definition G_v is called a *vertex group* and G_e is called an *edge group*.

For example, a generalized free product $G_1 *_H G_2$ can be associated with the graph \mathcal{G} with two distinct vertices v_1 and v_2 and an edge e joining them. The vertex groups are $G_{v_1} = G_1$ and $G_{v_2} = G_2$. The edge group is $G_e = H$ and the two injections $G_e \rightarrow G_{v_1}$ and $G_e \rightarrow G_{v_2}$ are the injections defining the generalized free product $G_1 *_H G_2$. In this case $G_1 *_H G_2$ is called the fundamental group of the graph \mathcal{G} . Similarly if $G *_H$ is an *HNN*-extension then the corresponding graph consists of a single vertex v and an edge e whose both end points are the same vertex. The following Figure describe the situation in detail.

Let \mathcal{G} be a finite graph of groups and $N_{\mathcal{G}} = (\text{number of vertices of } \mathcal{G} + \text{number of edges of } \mathcal{G})$. Now we proceed to define the *fundamental group* $\pi_1(\mathcal{G})$ of the finite graph of groups \mathcal{G} . At first when $N_{\mathcal{G}} = 1$ then the graph has one vertex v and no edge. Define $\pi_1(\mathcal{G}) = G_v$. By induction on $N_{\mathcal{G}}$ assume that we have defined $\pi_1(\mathcal{G})$ for all graph \mathcal{G} for which $N_{\mathcal{G}} \leq n - 1$. So let \mathcal{G} be a finite graph and $N_{\mathcal{G}} = n$. There are now two cases to consider.

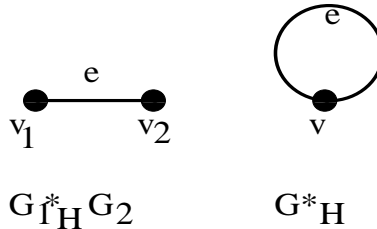


Figure: graph of groups corresponding to generalized free product and *HNN*-extension.

Case A. Let v be a vertex of \mathcal{G} with only one edge e emanating from it. Let G_v be the group associated to the vertex v . Let \mathcal{G}_1 be the graph obtained by deleting v and e from \mathcal{G} . Clearly \mathcal{G}_1 has $N_{\mathcal{G}_1} =$

$n - 2$. Hence by the induction hypothesis $\pi_1(\mathcal{G}_1)$ is defined. Now define $\pi_1(\mathcal{G}) = \pi_1(\mathcal{G}_1) *_{G_e} G_v$.

Case B. All the vertices have valency ≥ 2 , that is, there are at least two edges emanating from every vertex of the graph. Remove one edge e from the graph and let \mathcal{G}_1 be the resulting graph. Hence \mathcal{G}_1 has $N_{\mathcal{G}_1} = n - 1$ and by induction $\pi_1(\mathcal{G}_1)$ is defined. Then define $\pi_1(\mathcal{G}) = \pi_1(\mathcal{G}_1) *_{G_e}$.

To define the fundamental group of an infinite graph of groups \mathcal{G} note that \mathcal{G} can be written as an increasing union of finite subgraphs \mathcal{G}_i . Define $\pi_1(\mathcal{G}) = \lim_{i \rightarrow \infty} \pi_1(\mathcal{G}_i)$.

Remark 3.2. Note that in the above definition of fundamental group of a graph of groups there are many choices involved. But this definition does our purpose as any two choices define isomorphic groups.

The Reduction Theorem is for the simplest nontrivial graph, namely the graph with two vertices and one edge. The edge group is trivial and the vertex groups are G_1 and G_2 .

Corollary 3.2. *Let \mathcal{G} be a graph of groups with trivial edge groups. Also assume that the graph has countable number of vertices and edges and each vertex group is countable. If the FICwF is true for all vertex groups then FICwF is true for $\pi_1(\mathcal{G})$.*

4. STATEMENTS OF THEOREMS AND PROOFS OF BASIC RESULTS: THE FIC FOR 3-MANIFOLD GROUPS

To state the theorems we need to recall some more preliminaries from 3-manifold topology. An embedded closed 2-manifold F in a 3-manifold is called *2-sided* if the normal bundle of F in M is trivial, that is homeomorphic to $F \times \mathbb{R}$. For example any orientable 2-manifold embedded in an orientable 3-manifold is 2-sided. By a *Haken* 3-manifold we mean it is irreducible and contains an embedded 2-sided π_1 -injective closed 2-manifold of infinite fundamental group. Such an embedded 2-manifold is called an *incompressible* surface. Also we recall that there is a unique decomposition of any compact Haken 3-manifold by cutting the 3-manifold along finitely many incompressible tori so that each piece is either Seifert fibered or hyperbolic. This is known as JSJT (Jaco-Shalen, Johannson and Thurston) decomposition. Recall that a *graph manifold* is either a Seifert fibered space or is a Haken 3-manifold which has only Seifert fibered pieces in its JSJT decomposition or equivalently which is obtained by gluing along tori boundary components of finitely many Seifert fibered 3-manifolds.

Let us first state a theorem which is a consequence of some fundamental results by Farrell and Jones for manifolds of any dimension.

Theorem 4.1. ([8], [11]) *Let M be a closed nonpositively curved Riemannian manifold or a compact surface (may be with nonempty boundary). Then the FICwF is true for $\pi_1(M)$.*

Proof. In the closed manifold case the theorem is obtained by combining theorem 0.4 of [11] and the remark following it together with proposition 2.3 of [8]. In the closed surface case we also have to use Proposition 2.1. When the surface has nonempty boundary take the double N of M . Then $\pi_1(M) \wr K$ is a subgroup of $\pi_1(N) \wr K$ for any finite group K . Now using Lemma 2.1 and the previous case we complete the proof. \square

The following corollary to the above Theorem is crucial for this paper.

Corollary 4.1. *Let M be a closed Haken 3-manifold such that there is a hyperbolic piece in the JSJT decomposition of the manifold or let M be a compact irreducible 3-manifold with nonempty incompressible boundary and there is at least one torus boundary component. Then the FICwF is true for $\pi_1(M)$.*

Proof. In the closed case by [23] M supports a nonpositively curved Riemannian metric. Theorem 4.1 proves the corollary in this case.

Secondly let M be compact with incompressible boundary and there is at least one torus boundary. Let M' be the double of M along boundary components of genus ≥ 2 . Then M' is irreducible and with incompressible tori boundary components. Using [23] we deduce that the interior of M' supports a complete nonpositively curved Riemannian metric so that near the boundary components the metric is a product. Now let M'' be the double of M' . Then M'' supports a nonpositively curved Riemannian metric. Hence by the closed case the FICwF is true for $\pi_1(M'')$. On the other hand we have the inclusions $\pi_1(M) < \pi_1(M') < \pi_1(M'')$ which induce $\pi_1(M) \wr G < \pi_1(M') \wr G < \pi_1(M'') \wr G$. Here G is a finite group. Hence the FICwF is true for $\pi_1(M)$ by Lemma 2.1. \square

Another result we will be using often is the following theorem of Farrell and Linnell.

Theorem 4.2. ([12], theorem 7.1]) *Let I be a directed set, and let Γ_n , $n \in I$ be a directed system of groups with $\Gamma = \lim_{n \in I} \Gamma_n$; i.e., Γ is the direct limit of the groups Γ_n . If each group Γ_n satisfies the FIC, then Γ also satisfies the FIC.*

Let M be a complete Riemannian manifold. Then M is called *A-regular* if the following is satisfied.

$$|\nabla^i(R)| \leq A_i$$

where A_i is a sequence of nonnegative integers and $\nabla^i(R)$ denotes the i -th covariant derivative of the curvature tensor R of the Riemannian manifold M . For example; any compact or homogeneous or locally symmetric Riemannian manifold is *A-regular*. Some more examples, relevant for this article, of noncompact *A-regular* Riemannian manifolds are described in Section 10.

Definition 4.1. A group Γ is called an *A-group* if there is a complete simply connected *A-regular* Riemannian manifold M of nonpositive sectional curvature and a properly discontinuous action of Γ on M by isometries so that the image of Γ in $\text{Iso}(M)$ is virtually torsion free.

L.E. Jones recently proved that [[20], theorem 1.6] the assembly map in the statement of the Fibered isomorphism Conjecture induces a surjective homomorphism on the homotopy group level for torsion free *A-groups*. Note that the FIC states that this homomorphism should be an isomorphism. Note that *A-regularity* is a consequence if the action in the above Definition is also cocompact. Under this cocompactness assumption the FIC is proved in [8] for any *A-groups*, even if it has torsion (that is the Theorem 4.1).

Notation. Let us denote by \mathcal{C} the class of compact irreducible 3-manifolds with nonempty incompressible boundary so that each boundary component is a surface of genus ≥ 2 .

Definition 4.2. A group Γ is called a *B-group* if it contains a finite index subgroup isomorphic to the fundamental group of a member of \mathcal{C} .

In Proposition 4.1 we show that if $M \in \mathcal{C}$ then $\pi_1(M) \wr G$ is an *A-group* for any finite group G .

Though it is not yet proved that the FIC is true for all *A-groups*, we have some partial results in this direction.

Let M be a compact 3-manifold with nonempty boundary. A compact surface $F \subset M$ is called *properly embedded* if $\partial F \subset \partial M$ and $F - \partial F \subset M - \partial M$.

Definition 4.3. A properly embedded annulus A which is the image of an embedding $g : (F, \partial F) \rightarrow (M, \partial M)$ is called *essential* if the followings hold.

- the embedding g is not isotopic (relative to boundary) to an embedding $f : (F, \partial F) \rightarrow (M, \partial M)$ so that $f(F) \subset \partial M$.
- $g_* : \pi_1(F) \rightarrow \pi_1(M)$ is injective.

Remark 4.1. The above definition of essential embedding is little different from standard definition. Usually the second condition is not assumed in standard terminology and in the first condition it is only demanded that g is homotopic to a map $f : (F, \partial F) \rightarrow (M, \partial M)$ so that $f(F) \subset \partial M$.

In Section 9 we prove some results describing properties of members of \mathcal{C} . We show in Proposition 9.1 that, for any $M \in \mathcal{C}$, $\pi_1(M)$ is isomorphic to the fundamental group of a graph of groups with infinite cyclic edge groups and whose vertex groups are fundamental groups of members of \mathcal{C} which contain no essential annulus.

The following theorem is a first step towards proving the FIC for B -groups.

Theorem 4.3. *Let $M \in \mathcal{C}$. Assume that there is no essential annulus embedded in M . Then the FICwF is true for $\pi_1(M)$.*

The following Proposition describes the main set of examples of A -groups used in this article. We prove this Proposition in Section 10.

Proposition 4.1. *Let $M \in \mathcal{C}$. Then the followings are true.*

- *In the JSJT decomposition of M the pieces which contain a boundary component of M supports hyperbolic metric in the interior.*
- *The interior of M supports a nonpositively curved Riemannian metric so that near each boundary components the metric has constant -1 sectional curvature. Also the group $\pi_1(M) \wr G$ is an A -group for any finite group G .*

A second important step to prove the FIC for B -groups is the following.

Theorem 4.4. *Let $M \in \mathcal{C}$. Consider the Riemannian metric in the interior of M as given by Proposition 4.1. Assume that with respect to the hyperbolic metric near each boundary component the boundary components are totally geodesic. Then the FICwF is true for $\pi_1(M)$.*

In Examples 9.1 we show that there are members of \mathcal{C} which do not satisfy the hypothesis of Theorems 4.3 and 4.4. We do not know if these are the only such examples.

We are now ready to state the main result of this article.

Main Theorem. *Assume that the FIC is true for B -groups¹. Let M be a closed 3-manifold. Let H be a homomorphic image of $\pi_1(M)$ satisfying the following properties.*

- *H has a finite index nontrivial torsion free subgroup.*
- *the FICwF is true for H .*
- *any infinite cyclic subgroup of H has infinite index in H .*

Then the FICwF is true for $\pi_1(M)$.

There are several important consequences of the Main Theorem and its proof. Before we state the consequences we recall some background.

Recall that by Thurston's hyperbolization theorem if a Haken 3-manifold contains no incompressible torus then it is hyperbolic and hence in this case Theorem 4.1 applies. Thus in the class of Haken 3-manifolds, because of Corollary 4.1, the FIC remains to be proven for closed graph manifolds only.

The Geometrization conjecture implies that an irreducible 3-manifold either has finite fundamental group or it is one of the following types: Seifert fibered, hyperbolic or Haken.

By a *virtually fibered* 3-manifold we mean that it has a finite sheeted cover which fibers over the circle.

It is known that there are aspherical 3-manifolds which are not virtually fibered. But a well-known conjecture says that every irreducible 3-manifold with infinite fundamental group (and hence aspherical) has a finite sheeted cover which is Haken (in this case the manifold is called *virtually Haken*). This is called the *virtual Haken conjecture*.

Theorem 4.5. *Assume that the FIC is true for B -groups. Then the FICwF is true for Γ where Γ is isomorphic to the fundamental group of one of the following manifolds.*

- *compact 3-manifold with nonempty boundary or a noncompact 3-manifold.*
- *3-manifold which has a finite sheeted cover with first Betti number ≥ 2 .*
- *nontrivial graph manifold.*
- *virtually fibered 3-manifold.*
- *virtually Haken 3-manifold.*

Here we should point out that the different items in the above theorem are not mutually exclusive. But we state them separately for the importance and popularity of the individual classes of manifolds and groups.

In this connection we recall that in [33] we proved the following.

¹Recently in [34] we have shown that the FIC is true for B -groups.

Proposition 4.2. *[[33], proposition 7.1] Let M be a compact 3-manifold and there is a finite sheeted covering of M which fibers over the circle with special monodromy diffeomorphism. Then the FICwF is true for $\pi_1(M)$.*

For definition of special diffeomorphism of a surface see [[33], definition, p. 4].

Now we state some Theorems where we do not need the assumption that ‘the FIC is true for B -groups’ always. Important instances are Theorems 4.6 and 4.8.

Theorem 4.6. *Let S be a Seifert fibered space. Then the FICwF is true for $\pi_1(S)$.*

An important corollary of Corollary 3.1, Theorem 4.5 and Theorem 4.6 is the following.

Corollary 4.2. *Assume that the FIC is true for B -groups. Then the FICwF is true for any 3-manifold (irreducible or not) group if Thurston’s Geometrization conjecture is true.*

Below, in Theorem 4.8 we prove the FIC for some large classes of 3-manifolds and for a class of PD_3 groups in Theorem 4.7.

Note that we do not assume that M is a Haken 3-manifold in these theorems.

Recall that if M is a compact 3-manifold and there is a surjective homomorphism from $\pi_1(M)$ to \mathbb{Z} with finitely generated kernel then by Stallings’ fibration theorem M fibers over the circle and hence by Theorem 4.5 the FIC is true for $\pi_1(M)$ provided the FIC is true for any B -group. In fact here we deduce a more general statement showing that the fourth item of Theorem 4.5 is true for some general class of 3-dimensional Poincaré duality (PD_3) groups. This theorem is an application of Theorem 4.5 and [[18], theorem 1.20] and the well-known result that a PD_2 is a surface group ([5] and [6]).

Theorem 4.7. *Assume that the FIC is true for any B -group. Let \mathcal{G} be a group and H a finite index, finitely presented subgroup of \mathcal{G} . Assume that H is a PD_3 group and there is a surjective homomorphism from H to \mathbb{Z} with finitely generated kernel. Then the FICwF is true for \mathcal{G} .*

It is known that any nontrivial graph manifold has a finite sheeted cover which either fibers over the circle or has large first Betti number (see [24]). We have already considered the case when M is a virtual fiber bundle. Hence we are left with the second case when there is a finite sheeted cover with a large first Betti number.

Next we use the technique used in the proof of [[33], main lemma] to prove the FIC for another class of 3-manifold groups.

Let M be a closed 3-manifold.

Definition 4.4. We say M satisfies *Condition** if it satisfies the following two conditions.

- rank of $H_1(M, \mathbb{Z})$ is ≥ 1 ,
- either
 - (a). $[\pi_1(M), \pi_1(M)]$ is finitely generated or
 - (b). M is a nontrivial graph manifold, rank of $H_1(M, \mathbb{Z})$ is greater than 1, and $[\pi_1(M), \pi_1(M)] \cap \pi_1(P)$ is not a free group for any Seifert fiber piece P in the JSJT decomposition of M .

Remark 4.2. In the category of graph of groups, (b) in Condition* has the following rewording. Note that $\pi_1(M)$ has the structure of a graph of groups where each vertex represents a Seifert fibered piece in M and the associated group is the fundamental group of the Seifert fibered piece. An edge is represented by a gluing incompressible torus. As a subgroup of $\pi_1(M)$, $[\pi_1(M), \pi_1(M)]$ inherits a graph of group structure. (b) in Condition* says that the groups associated to each of the vertices of this graph are not free. Also (b) together with [[16], theorem 11.1] implies that $[\pi_1(M), \pi_1(M)]$ is infinitely generated. This follows from the fact that if the commutator subgroup is finitely generated and infinite cyclic then M is a Seifert fibered space which is a contradiction because M is a nontrivial graph manifold. On the other hand if the commutator subgroup is finitely generated and not infinite cyclic then (2) of [[16], theorem 11.1] implies that rank of $H_1(M, \mathbb{Z})$ is 1 which is again a contradiction.

A class of examples which satisfy Condition* are described in Section 11.

Theorem 4.8. *Let M be a closed 3-manifold such that there is a finite sheeted regular cover \tilde{M} of M satisfying Condition*. When $[\pi_1(\tilde{M}), \pi_1(\tilde{M})]$ is finitely generated assume that the FIC is true for B -groups. Then the FICwF is true for $\pi_1(M)$.*

Remark 4.3. Notice that when $[\pi_1(\tilde{M}), \pi_1(\tilde{M})]$ is infinitely generated then we do not need the assumption that ‘the FIC is true for B -groups’. Also when $[\pi_1(\tilde{M}), \pi_1(\tilde{M})]$ is finitely generated then one can deduce (using Proposition 8.1) that \tilde{M} is finitely covered by a 3-manifold fibering over the circle.

In this connection we pose the following problem.

Problem. Let M be a closed graph manifold. Then show that M satisfies Condition*.

Now let us recall that the fourth item of Theorem 4.5 dealt with 3-manifolds which are virtually of the type as in Stallings' fibration theorem. The following result consider a similar but more general class of 3-manifold.

Theorem 4.9. *Let M be a compact 3-manifold so that there is a finite index normal subgroup G of $\pi_1(M)$ which sits in the following exact sequence*

$$1 \rightarrow H \rightarrow G \rightarrow Q \rightarrow 1$$

where H is finitely generated and Q is an infinite group. If H is not infinite cyclic then assume that the FIC is true for any B -group. Then the FICwF is true for $\pi_1(M)$.

The following corollary is an immediate consequence of the results we have proved.

Corollary 4.3. *Let G be any of the groups for which we have proved the FICwF, then the Whitehead group $Wh(H)$, the reduced projective class group $\tilde{K}_0(H)$ and $K_{-i}(H)$ for $i \geq 1$ vanish for any torsion free subgroup H of G .*

The above corollary gives a partial solution to problem 3.32 of the R. Kirby's problem list in [22]. See [[22], p. 168] for a general discussion about known results related to [[22], problem 3.32]. However, modulo Geometrization Conjecture, Corollary 4.3 is true for any 3-manifold M , whenever $\pi_1(M)$ is torsion free. Also recall that $Wh(G) = \tilde{K}_0(G) = K_{-i}(G) = 0$ for $i \leq 1$ is a classical result of Waldhausen for a Haken 3-manifold group G (see [36]) as mentioned in the introduction.

5. PROOFS OF THE REDUCTION THEOREM AND ITS' CONSEQUENCES

For notations and conventions used in this section we refer the reader to [4]. The technique used are from the theory of action of groups on trees and the main result we use is the structure theorem of groups acting on trees.

Proof of the Reduction Theorem. Note that there is a surjective homomorphism $\pi : G_1 * G_2 \rightarrow G_1 \times G_2$ and by the following Lemma the FIC is true for $G_1 \times G_2$.

Lemma 5.1. *Let G_1 and G_2 be two groups and assume that the FIC is true for both G_1 and G_2 then the FIC is true for the product $G_1 \times G_2$.*

Proof. Consider the projection $p_1 : G_1 \times G_2 \rightarrow G_1$. By Lemma 2.2 we need to check that the FIC is true for $p_1^{-1}(C)$ for any virtually cyclic subgroup C of G_1 . Note that $p_1^{-1}(C) = C \times G_2$. Now consider the projection $p_2 : C \times G_2 \rightarrow G_2$. Again we apply Lemma 2.2. That is we need to show that the FIC is true for $p_2^{-1}(C')$ for any virtually cyclic subgroup C' of G_2 . But $p_2^{-1}(C') = C \times C'$ which is virtually poly- \mathbb{Z} and hence the FIC is true for $p_2^{-1}(C')$. This completes the proof of the Lemma. \square

Thus (by Lemma 2.2) it is enough to check that the FIC is true for $\pi^{-1}(C)$ for any virtually cyclic subgroup C of $G_1 \times G_2$. We need the following Lemma.

Lemma 5.2. *For a subgroup C of $G_1 \times G_2$ the followings are true.*

- $\pi^{-1}(C)$ is a countable free group if C is either the trivial group or the infinite cyclic group.
- $\pi^{-1}(C)$ contains a countable free subgroup of finite index if C is a finite group.

Proof. Let us first recall that the free product $G_1 * G_2$ acts on a tree \mathcal{T} with vertex stabilizers gG_1g^{-1} or gG_2g^{-1} , where g varies in $G_1 * G_2$, and trivial edge stabilizers. Hence any subgroup H of $G_1 * G_2$ also acts on \mathcal{T} with vertex stabilizers $gG_1g^{-1} \cap H$ or $gG_2g^{-1} \cap H$, where $g \in G_1 * G_2$. Now it is easy to check that $\ker(\pi) \cap gG_i g^{-1}$ is trivial for $i = 1, 2$ and for any $g \in G_1 * G_2$. Hence using the structure theorem of groups acting on trees [[4], I.3.4] it follows that $\ker(\pi)$ is a free group. Now let C be an infinite cyclic subgroup of $G_1 \times G_2$. Then one checks that $\pi^{-1}(C) \cap gG_i g^{-1}$ is either infinite cyclic or the trivial group for $i = 1, 2$ and for $g \in G_1 * G_2$. Hence again we appeal to [[4], I.3.4] to deduce that $\pi^{-1}(C)$ is a free group.

The second assertion is an immediate consequence of the first. \square

Let C be a virtually cyclic subgroup of $G_1 \times G_2$. If C is finite then $\pi^{-1}(C) < \ker(\pi) \wr C$ and if C is infinite and C' is an infinite cyclic normal subgroup of C of finite index then $\pi^{-1}(C) < \pi^{-1}(C') \wr (C/C')$. In both the cases, by Lemma 5.2 we get that $\pi^{-1}(C)$ is a subgroup of $F \wr H$ where F is a free group and H is a finite group. The proof now follows from the following lemma.

Lemma 5.3. *The FICwF is true for F where F is a countable free group.*

Proof. If F is finitely generated then by Theorem 4.1 the FICwF is true for F . If F is infinitely generated then let $F \simeq \lim_{i \rightarrow \infty} F_i$ where

F_i is a finitely generated free group. We get $F \wr H < \lim_{i \rightarrow \infty} (F_i \wr H)$. Now we appeal to Theorem 4.2 and Theorem 4.1 to deduce that the FICwF is true for F . \square

Using Lemma 2.1 we complete the proof of the Reduction Theorem. \square

Proof of Theorem 3.1. Let L be a finite group. Note that there is a surjective homomorphism $p : (G_1 * G_2) \wr L \rightarrow (G_1 \times G_2) \wr L$. It is easy to see that $(G_1 \times G_2) \wr L$ is a subgroup of $G_1 \wr L \times G_2 \wr L$. By hypothesis and by Lemma 5.1 the FIC is true for $(G_1 \wr L) \times (G_2 \wr L)$ and hence for $(G_1 \times G_2) \wr L$ also. Kernel of p is $(\ker(\pi))^L$, where π is the surjective homomorphism $G_1 * G_2 \rightarrow G_1 \times G_2$, that is $\ker(p)$ is a direct product of $|L|$ copies of $\ker(\pi)$. Let C be a virtually cyclic subgroup of $(G_1 \times G_2) \wr L$ and C' be a cyclic subgroup of $(G_1 \times G_2)^L \cap C$ which is normal and of finite index in C . Let C' be generated by $\gamma = (\gamma_1, \dots, \gamma_k)$ where $|L| = k$. Then the action of γ on $(G_1 * G_2)^L$ is factorwise, that is the i -th coordinate of γ acts on the i -th factor of $(G_1 * G_2)^L$. Without loss of generality we can assume that $\gamma_i \neq 1$ for $i = 1, \dots, l$ and $\gamma_i = 1$ for $i = l+1, \dots, k$. Hence $p^{-1}(C')$ is a subgroup of $\pi^{-1}(\langle \gamma_1 \rangle) \times \dots \times \pi^{-1}(\langle \gamma_l \rangle) \times (\pi^{-1}(1))^{k-l} = H$ (say). By Lemma 5.2 each of the factor in the above expression is either free or contains a free subgroup of finite index. Now we have

$$\begin{aligned} p^{-1}(C) &< p^{-1}(C') \wr (C/C') < H \wr (C/C') \\ &< \pi^{-1}(\langle \gamma_1 \rangle) \wr (C/C') \times \dots \times \pi^{-1}(\langle \gamma_l \rangle) \wr (C/C') \times (\pi^{-1}(1) \wr (C/C'))^{k-l}. \end{aligned}$$

Using Lemma 5.3 and the following easily verified Lemma we complete the proof of the Theorem.

Lemma 5.4. *Let A and B be finite groups and G is any group, then $(G \wr A) \wr B$ is a subgroup of $G^{A \times B} \wr (A \wr B)$*

\square

Proof of Corollary 3.1. At first note that the FICwF is true for finite groups and a compact irreducible 3-manifold is either aspherical or has finite fundamental group. Therefore, for compact 3-manifolds the Corollary follows from Lemma 3.1, Theorem 3.1 and Lemma 5.3. If the 3-manifold is noncompact then we can write the manifold as an increasing union (under inclusion) of compact submanifolds. Now the proof follows from the previous case and by Theorem 4.2. \square

Proof of Corollary 3.2. Let us first prove the Corollary for finite graph of groups. Recall the notation $N_{\mathcal{G}}$ denoting the sum of number of vertices and number of edges of a finite graph \mathcal{G} . The proof is by

induction on $N_{\mathcal{G}}$. If $N_{\mathcal{G}} = 1$ then there is only one vertex and no edge and hence FICwF is true for $\pi_1(\mathcal{G})$ by hypothesis. Assume that the Corollary is true for finite graphs with $N_{\mathcal{G}} \leq n - 1$. Let \mathcal{G} be a graph with $N_{\mathcal{G}} = n$. There are now two cases to consider.

A. Let v be a vertex of \mathcal{G} with only one edge emanating from it. Let G_v be the groups associated to the vertex v . Then, clearly $\pi_1(\mathcal{G}) \simeq \pi_1(\mathcal{G}_1) * G_v$ where \mathcal{G}_1 is a graph with $N_{\mathcal{G}_1} = n - 2$. Hence by the induction hypothesis and by Theorem 3.1 FICwF is true for $\pi_1(\mathcal{G})$.

B. All the vertices have valency ≥ 2 . Remove an edge from the graph and let \mathcal{G}_1 be the resulting graph. Then clearly $\pi_1(\mathcal{G}) \simeq \pi_1(\mathcal{G}_1) * \mathbb{Z}$. Again by induction hypothesis and Theorem 3.1 the proof is completed in the finite graph case.

Now we consider the infinite graph case. Note that the fundamental group of any infinite graph which has countable number of vertices and edges can be written as a direct limit of fundamental groups of an increasing sequence of finite subgraphs. Using Theorem 4.2 and the previous case we complete the proof of the Corollary. \square

6. PROOFS OF THEOREMS 4.3, 4.4 AND THE MAIN THEOREM

Proof of Theorem 4.3. Let N be the double of M . Then N is a closed Haken 3-manifold. If N is a Seifert fibered space then the FICwF is true for $\pi_1(N)$ by Theorem 4.6. Also since $\pi_1(M)$ is a subgroup of $\pi_1(N)$, FICwF is true for $\pi_1(M)$. So assume that N is not Seifert fibered. By Corollary 4.1 we can also assume that N is not hyperbolic. Consider the JSJT decomposition of M . This says that there is a collection of finitely many embedded incompressible tori (say \mathcal{T}), unique up to ambient isotopy, in M so that the complementary pieces are either Seifert fibered or supports a complete hyperbolic metric.

The proof of the Theorem will be completed using the following two crucial lemmas.

Lemma 6.1. *The family of tori \mathcal{T} can be isotoped in N to be disjoint from $\partial M \subset N$.*

Proof. Assume on the contrary, that is, there is a member $T \in \mathcal{T}$ which can not be isotoped off ∂M . Choose an isotope T' of T which intersects ∂M transversally so that the number of circles in $C = \partial M \cap T'$ is minimal. We can assume that no circle in C bounds a 2-disc on T' . To see this note that if there is such a disc then we can isotope T' further to push it off from ∂M making the number of circles in C one less. Also since T cannot be isotoped off ∂M and since $N - \partial M$ is disconnected there are more than one circles in C . Let A be a component of $\overline{T' - C}$.

Then A is a properly embedded essential annulus in M . Which is a contradiction. This proves the lemma. \square

Lemma 6.2. *N is not a graph manifold.*

Proof. At first by Lemma 6.1 isotope \mathcal{T} to make it disjoint from ∂M . Let F be a component of ∂M . Assume N is a graph manifold and choose a Seifert fibered piece S in N in the JSJT decomposition of N so that $F \subset S$. Since N is not a Seifert fibered space we conclude that $\partial S \neq \emptyset$. Thus S is a Seifert fibered space with nonempty boundary and hence it has a finite sheeted cover which is a product of a surface F' and the circle. Note that $\partial F' \neq \emptyset$. Since F is an incompressible surface in S we get $\pi_1(F) \cap (\pi_1(F') \times \mathbb{Z})$ is a finite index subgroup of $\pi_1(F)$. On the other hand either $\pi_1(F) \cap (\pi_1(F') \times \mathbb{Z})$ is free or $\pi_1(F)$ contains a free abelian subgroup of rank 2. Which is a contradiction, since F is a closed surface of genus ≥ 2 . Hence N is not a graph manifold. \square

The proof of the theorem is now easy. Since N is a Haken 3-manifold and by Lemma 6.2 it is not a graph manifold, it follows that there is a hyperbolic piece in the JSJT decomposition of N and hence by Corollary 4.1 FICwF is true for $\pi_1(N)$. Consequently for $\pi_1(M)$ as well.

This completes the proof of the Theorem. \square

Proof of Theorem 4.4. Let M_1 be a piece of M in the JSJT decomposition of M so that $\partial M \cap \partial M_1 \neq \emptyset$. Then by Proposition 4.1 the interior of M_1 supports a complete hyperbolic metric. Since by hypothesis all the boundary components of M are totally geodesic, if M_2 is the double of M_1 along the boundary components of genus ≥ 2 , then the interior of M_2 also supports a complete hyperbolic Riemannian metric. Now let N be the double of M . Then $M_2 \subset N$ and in the JSJT decomposition of N , M_2 is a hyperbolic piece. Hence by [23], N supports a nonpositively curved Riemannian metric. Consequently the FICwF is true for $\pi_1(N)$. By Lemma 2.1 we complete the proof of the Theorem. \square

From the proof of Theorems 4.3 and 4.4 we deduce the following corollary.

Corollary 6.1. *Let $M \in \mathcal{C}$ and M does not contain any properly embedded essential annulus or assume that each boundary component is totally geodesic with respect to the metric as described in Proposition 4.1. Then $\pi_1(M)$ is a subgroup of the fundamental group of a closed 3-manifold P so that P is either Seifert fibered or is a nonpositively curved Riemannian 3-manifold.*

Proof of the Main Theorem. By hypothesis we have the following exact sequence.

$$1 \rightarrow K \rightarrow \pi_1(M) \rightarrow H \rightarrow 1.$$

Case A. Let us assume that H is torsion free.

At first we check that the FIC is true for $\pi_1(M)$.

Let M_L denote the covering of M corresponding to a subgroup L of $\pi_1(M)$. When L has infinite index then M_L is a noncompact 3-manifold. Since the FIC is true for H , by Lemma 2.2 we have to check that the FIC is true for $p^{-1}(C)$ for any virtually cyclic subgroup (including trivial) C of H . Here p is the homomorphism $\pi_1(M) \rightarrow H$. Since H is torsion free C is either trivial or infinite cyclic.

By the last assumptions in the statement of the theorem $M_{p^{-1}(C)}$ is a noncompact 3-manifold. Hence by choosing a proper smooth map from $M_{p^{-1}(C)}$ to the real line we can write $M_{p^{-1}(C)}$ as a union of increasing sequence of nonsimply connected, connected, compact submanifolds M_C^i with nonempty boundary. Hence

$$p^{-1}(C) \simeq \lim_{i \rightarrow \infty} \pi_1(M_C^i).$$

Since cover of an orientable irreducible 3-manifold is irreducible, $M_{p^{-1}(C)}$ is irreducible (see theorem 3, section 7 of [27]). Therefore each M_C^i has at least one boundary component of genus ≥ 1 . To see this we state and prove the following Lemma.

Lemma 6.3. *Let M be a nonsimply connected irreducible 3-manifold which is either noncompact or compact with nonempty boundary. Let N be a compact connected nonsimply connected 3-dimensional submanifold of M . Then there is at least one boundary component of N of genus ≥ 1 .*

Proof. On the contrary assume that all the boundary components of N are spheres, say S_1, S_2, \dots, S_k . Since M is irreducible there is a 3-disc $D_1 \subset M$ so that $\partial D_1 = S_1$. Again since M is irreducible by [[16], lemma 3.8] $M - S_1$ is disconnected. Let $M - S_1 = M_1 \cup \text{int}(D_1)$ where M_1 is connected. Since N is connected either $N \subset D_1$ or $N \subset \overline{M_1}$. In the former case it is easy to show that N is simply connected. Which is not possible by hypothesis. So assume $N \subset \overline{M_1}$. Let $N_1 = N \cup D_1$. Then N_1 is a submanifold of M with one less boundary components. Continuing this process we find a closed submanifold (after capping off all the boundary components of N) of M of dimension 3 - which is a contradiction. \square

By Theorem 4.2 it is enough to check that the FIC is true for $\pi_1(M_C^i)$. By capping off all the sphere boundary components of M_C^i we can

assume that all the boundary components of M_C^i are surfaces of genus ≥ 1 .

We need the following lemma.

Lemma 6.4. $\pi_1(M_C^i) \simeq G_1 * \cdots * G_k * F^r$ where F^r is a free group of rank r and each G_i is isomorphic to the fundamental group of a compact irreducible 3-manifold with incompressible boundary.

Proof. By Lemma 3.1 $\pi_1(M_C^i)$ is isomorphic to a free product of finitely many compact irreducible 3-manifold (with nonempty boundary of genus ≥ 1 by Lemma 6.3) groups and a free group of finite rank. We would like to use the Loop Theorem of Papakyriakopolous to reduce the irreducible 3-manifolds to irreducible 3-manifolds with nonempty incompressible boundary components.

We recall the Loop Theorem below.

The Loop Theorem. ([16], chapter 4, the loop theorem) *Let M be a compact 3-manifold with boundary ∂ . Let γ be a nontrivial element of $\pi_1(\partial)$ which goes to the trivial element in $\pi_1(M)$. Then there is a simple closed curve on ∂ which represents a nontrivial element in $\pi_1(\partial)$ and bounds a properly embedded disc in M .*

Let N be a compact connected nonsimply connected irreducible 3-manifold with nonempty boundary. Then all the boundary components of N are surfaces of genus ≥ 1 . Let F be a boundary component of N so that the kernel of $\pi_1(F) \rightarrow \pi_1(N)$ is nontrivial. Then using the Loop theorem we get an properly embedded disc D in N . We cut N along D to get a new 3-manifold $N' \subset N$ with boundary. Using Lemma 6.3 we get that either both the components (if it is disconnected) of N' are simply connected or there is at least one boundary component of genus ≥ 1 . If components of N' are simply connected then we stop here. Otherwise we cap off all the sphere boundary components by 3-discs and denote it by N' again. If N' has two components N_1 and N_2 then $\pi_1(N) \simeq \pi_1(N_1) * \pi_1(N_2)$. And if N' has one component then $\pi_1(N) \simeq \pi_1(N') * \mathbb{Z}$. Now if some boundary component of N' is compressible we apply again the Loop theorem. Standard technique in 3-manifold topology asserts that this process stops at finite stage. That is after finitely many application of the Loop theorem we will get finitely many 3-manifolds which are either simply connected or (after capping off sphere boundary components by 3-discs) compact 3-manifolds with nonempty boundary and each boundary component is incompressible. If any of these 3-manifolds is not irreducible then we apply Lemma 3.1 again.

This completes the proof of the Lemma. □

By the Reduction Theorem and Theorem 4.1 we only have to check that the FIC is true for G_i for each i . So let N be a compact irreducible 3-manifold with incompressible boundary components. If at least one of the boundary component is a torus then by Corollary 4.1 the FICwF is true for $\pi_1(N)$. So assume that all the components of ∂N are surfaces of genus greater or equal to 2. In this case by hypothesis the FICwF is true for $\pi_1(N)$. Now the proof of the Theorem follows from Lemma 6.4 and Theorem 3.1

This completes the proof of the Main Theorem in **Case A**.

Case B. Let J be a torsion free normal subgroup of H of finite index. Then $\pi_1(M)$ is a subgroup of $p^{-1}(J) \wr (H/J)$. Note that $p^{-1}(J)$ is again the fundamental group of a closed 3-manifold so that there is an exact sequence $1 \rightarrow K \rightarrow p^{-1}(J) \rightarrow J \rightarrow 1$ satisfying all the three properties in the statement of the Main Theorem and of **Case A**.

Using Lemma 5.4 we see that to complete the proof of the Main Theorem we only have to check that the FICwF is true for $\pi_1(M^k)$ where M is as in the statement with H torsion free and k is a positive integer.

Now we are in a more general setting as described in the following two Lemmas.

Lemma 6.5. *Assume that the FICwF is true for H . Then the FICwF is true for H^k for any positive integer k .*

Proof. The proof follows from Lemma 5.1 and by noting that $(H \times H) \wr G$ is a subgroup of $(H \wr G) \times (H \wr G)$. Here G is a finite group. \square

Lemma 6.6. *Let K be a normal subgroup of a group Γ with quotient group Q and $A : \Gamma \rightarrow Q$ is the quotient homomorphism. Assume the following.*

- $A^{-1}(Z) \simeq \lim_{i \rightarrow \infty} \Gamma_i^Z$ where Z is either trivial or an infinite cyclic subgroup of Q and $\{\Gamma_i^Z\}$ is a directed system of groups so that for each i the FICwF is true for Γ_i^Z .
- the FICwF is true for Q .

Then the FICwF is true for Γ .

Before we give the proof of Lemma 6.6 let us first complete the proof of the Main theorem using the Lemma. We only have to check that the hypothesis of the above Lemma is satisfied. We have already seen that $p^{-1}(C) \simeq \lim_{i \rightarrow \infty} \pi_1(M_C^i)$ and $\pi_1(M_C^i) \simeq G_1 * \cdots * G_k * F^r$ where F^r is a free group and each G_i is the fundamental group of a compact irreducible 3-manifold with incompressible boundary. Set $K = K, p^{-1}(J) = \Gamma$ and $Q = J$. Then using Theorem 3.1, Theorem

4.1, Corollary 4.1 and **Case A** we see that the hypothesis of the Lemma is satisfied.

Thus we have completed the proof of the Main Theorem. \square

Proof of Lemma 6.6. Let L be a finite group. We have the following exact sequence.

$$1 \rightarrow K^L \rightarrow \Gamma \wr L \rightarrow Q \wr L \rightarrow 1.$$

Here K is the kernel of A .

We denote the map $\Gamma \wr L \rightarrow Q \wr L$ also by A .

By hypothesis the FIC is true for $Q \wr L$. Since $K \simeq \lim_{i \rightarrow \infty} \Gamma_i^{(1)}$ and the FIC is true for each $\Gamma_i^{(1)}$, by Theorem 4.2 the FIC is true for K and hence for the product K^L by Lemma 5.1.

Let Z' be a virtually cyclic subgroup of $Q \wr L$. If Z' is finite then $A^{-1}(Z') < K^L \wr Z' < \lim_{i \rightarrow \infty} ((\Gamma_i^{(1)} \wr Z') \times (\Gamma_i^{(1)} \wr Z') \times \cdots \times (\Gamma_i^{(1)} \wr Z'))$. There are $|L|$ number of factors inside the bracket. Hence applying Theorem 4.2, Lemma 5.1 and Lemma 2.1 we get that the FIC is true for $A^{-1}(Z')$. If Z' is infinite let Z be the intersection of Z' with the torsion free part of $Q^L < Q \wr L \simeq Q^L \rtimes L$. Then $A^{-1}(Z') < A^{-1}(Z) \wr Z'/Z$. On the other hand $A^{-1}(Z) \simeq K^L \rtimes Z$ where the action of Z on K^L is factorwise. That is the j -th coordinate of a generator of $Z < Q^L$ acts on the j -th factor of K^L . Hence

$$K^L \rtimes Z < (K \rtimes Z_1) \times \cdots \times (K \rtimes Z_{|L|}).$$

Here Z_j is generated by the j -th coordinate of a generator of Z . The right hand side of the above expression is a subgroup of

$$\begin{aligned} & (\lim_{i \rightarrow \infty} \Gamma_i^{Z_1}) \times \cdots \times (\lim_{i \rightarrow \infty} \Gamma_i^{Z_{|L|}}) \\ & < \lim_{i \rightarrow \infty} (\Gamma_i^{Z_1} \times \cdots \times \Gamma_i^{Z_{|L|}}). \end{aligned}$$

Hence

$$\begin{aligned} A^{-1}(Z') & < A^{-1}(Z) \wr Z'/Z < \lim_{i \rightarrow \infty} (\Gamma_i^{Z_1} \times \cdots \times \Gamma_i^{Z_{|L|}}) \wr Z'/Z \\ & < \lim_{i \rightarrow \infty} ((\Gamma_i^{Z_1} \times \cdots \times \Gamma_i^{Z_{|L|}}) \wr Z'/Z) \\ & < \lim_{i \rightarrow \infty} ((\Gamma_i^{Z_1} \wr Z'/Z) \times \cdots \times (\Gamma_i^{Z_{|L|}} \wr Z'/Z)). \end{aligned}$$

By hypothesis the FIC is true for each of the factors inside the limit and hence for $A^{-1}(Z')$ also.

This proves the lemma. \square

7. PROOF OF THEOREM 4.5

Proof. We give the proof of the theorem item wise.

- At first we consider the noncompact case. The compact case will follow from it. So let M be a noncompact 3-manifold. Then by choosing a proper smooth map from M to the real line we can write M as a union of increasing sequence (under inclusion) of compact submanifolds M_i with nonempty boundary. By Lemmas 6.3 and 6.4 $\pi_1(M_i) \simeq G_1 * \cdots * G_k * F^r$ where each G_i is isomorphic to the fundamental group of a compact irreducible 3-manifold with incompressible boundary. The proof now follows by combining Theorem 4.2, Theorem 4.1 and Corollary 4.1.
- By the previous item we can assume that the manifold M is closed. Let N be a finite sheeted cover of M with first Betti number ≥ 2 . Then $\pi_1(M)$ is a subgroup of $\pi_1(N) \wr G$ for some finite group G . If we put $H = H_1(N, \mathbb{Z})$ then we check that all the three properties in the statement of the Main Theorem are satisfied. Being a finitely generated abelian group, obviously H has a torsion free subgroup of finite index. $H \wr G$ is a virtually poly- \mathbb{Z} group and hence the FIC is true for $H \wr G$ (Proposition 2.1) for any finite group G . Since H has rank greater or equal to 2 the third condition is satisfied. Hence by the Main Theorem the FICwF is true for $\pi_1(N)$.
- Let M be a nontrivial graph manifold. That is there is an incompressible torus embedded in the manifold. Now we appeal to [24] where it is proved that in this situation either M has a finite sheeted cover which is a torus bundle over the circle or there is a finite sheeted cover of M with arbitrarily large first Betti number. In the first case it follows that $\pi_1(M)$ is virtually poly- \mathbb{Z} for which the FIC is true (Proposition 2.1). In the second case the result follows from the last item.

For the second assertion recall that Thurston's Geometrization Conjecture claims that any compact irreducible 3-manifold either has a finite fundamental group or is Seifert fibered, Haken or hyperbolic. Using Theorem 4.6 we complete the proof of this item.

- By [[13], algebraic lemma] and Lemma 6.5 it is enough to prove the FICwF for any fibered 3-manifold. So let M be a 3-manifold fibering over the circle. Let F be the fiber of the fiber bundle projection $M \rightarrow \mathbb{S}^1$. If F is the sphere, torus or the Klein

bottle then $\pi_1(M) \wr G$ is virtually poly- \mathbb{Z} and hence the proof is complete in this case. So assume the F has genus ≥ 2 .

Since the commutator subgroup is characteristic we have the following exact sequence.

$$1 \rightarrow [\pi_1(F), \pi_1(F)] \rightarrow \pi_1(M) \rightarrow H_1(F, \mathbb{Z}) \rtimes \langle t \rangle \rightarrow 1.$$

Let $H = H_1(F, \mathbb{Z}) \rtimes \langle t \rangle$. Then it is easy to check all the three properties in the statement of the Main Theorem for the group H .

- The proof of this case follows from the proof of the third item.

□

8. PROOFS OF THEOREMS 4.6, 4.7, 4.8 AND 4.9

Proof of Theorem 4.6. If $\pi_1(S)$ is finite then it is known that the FICwF is true for this group. So assume that $\pi_1(S)$ is infinite. Then there exists the following well-known exact sequence.

$$1 \rightarrow C \rightarrow \pi_1(S) \rightarrow \pi_1^{orb}(B) \rightarrow 1.$$

Here $\pi_1^{orb}(B)$ denotes the orbifold fundamental group of the base orbifold of the Seifert fibered space S and C is an infinite cyclic group generated by a regular fiber of the Seifert fibration of S . Let L be a finite group and consider the following exact sequence.

$$1 \rightarrow C^L \rightarrow \pi_1(S) \wr L \rightarrow \pi_1^{orb}(B) \wr L \rightarrow 1.$$

Note that when B is closed, $\pi_1^{orb}(B)$ contains a normal closed surface subgroup of finite index and hence by Theorem 4.1 and Lemma 6.5 the FICwF is true for $\pi_1^{orb}(B)$. In the compact with nonempty boundary case take the double of B and apply Lemma 2.1. Let $p : \pi_1(S) \wr L \rightarrow \pi_1^{orb}(B) \wr L$ be the projection map and V is a virtually cyclic subgroup of $\pi_1^{orb}(B) \wr L$. Then $p^{-1}(V)$ is virtually poly- \mathbb{Z} and hence the FIC is true for $p^{-1}(V)$ by Proposition 2.1. Thus by Lemma 2.2 the FIC is true for $\pi_1(S) \wr L$. □

Proof of Theorem 4.7. If the kernel K of the homomorphism $H \rightarrow \mathbb{Z}$ is finite then \mathcal{G} is virtually cyclic and hence the FIC is true for \mathcal{G} . So we assume that K is finitely generated and infinite. By [[18], theorem 1.20] K is a PD_2 group and hence by [[5], [6]] K is isomorphic to the fundamental group of a closed surface of positive genus. Since any automorphism of a closed surface is induced by a diffeomorphism of the surface it follows that H is isomorphic to the fundamental group of a closed 3-manifold L which fibers over \mathbb{S}^1 . Thus $\mathcal{G} < \pi_1(L) \wr U$ where U is a finite group. By Theorem 4.5 we complete the argument. □

We need the following lemma to prove Theorem 4.8.

Lemma 8.1. *Let M be a compact graph manifold and Γ is a normal subgroup of $\pi_1(M)$ of infinite index and with the property that $\Gamma \cap \pi_1(P)$ is not free for any Seifert fibered piece P in the JSJT decomposition of M . Let M_Γ be the covering of M corresponding to Γ . Then $M_\Gamma = \cup_i N_i$ where for each $i = 1, 2, \dots$, $N_i \subset N_{i+1}$ and N_i is a compact irreducible 3-manifold with incompressible tori boundary components.*

Proof. Let $f : M_\Gamma \rightarrow M$ be the covering projection and let P be a Seifert fibered piece of M . Let \tilde{P} be a component of $f^{-1}(P)$. We will show that \tilde{P} is a Seifert fibered space. Note that M_Γ is noncompact and has empty boundary. Noncompactness implies there is a smooth proper surjective function $\delta : M_\Gamma \rightarrow \mathbb{R}^{\geq 0}$ or $\delta : M_\Gamma \rightarrow \mathbb{R}$ according as M_Γ has 1 or more than 1 ends respectively.

The proof of the lemma now consists of similar ideas as in the proof of [[33], corollary 6.2]. At first recall that there is a central infinite cyclic subgroup Z of $\pi_1(P)$ with quotient $\pi_1^{orb}(B_P)$, where B_P is the base orbifold of P ([16]). We claim that $Z \cap (\Gamma \cap \pi_1(P))$ is nontrivial. If not then $\Gamma \cap \pi_1(P)$ injects into $\pi_1^{orb}(B_P)$. Since B_P has nonempty boundary and $\Gamma \cap \pi_1(P)$ is torsion free we get that $\Gamma \cap \pi_1(P)$ is free - which is a contradiction. Now since $f_*(\pi_1(\tilde{P}))$ is conjugate to $\Gamma \cap \pi_1(P)$, $\pi_1(\tilde{P})$ contains an infinite cyclic central subgroup. Hence we can apply [[26], theorem 1.1] to deduce that \tilde{P} is Seifert fibered.

Now we proceed to give a filtration of \tilde{P} in an increasing union of compact irreducible 3-manifolds with incompressible tori boundary components. Let $B_{\tilde{P}}$ be the base surface of \tilde{P} . Choose a filtration of $B_{\tilde{P}}$ by increasing union of compact subsurfaces with incompressible circular boundary components so that the boundary components do not contain any of the orbifold points. Now pulling back this filtration by the quotient map $\tilde{P} \rightarrow B_{\tilde{P}}$ we get a filtration of \tilde{P} by compact irreducible 3-manifolds \tilde{P}_i with incompressible tori boundary components. Thus we get a covering of M_Γ by the collection $\{\tilde{P}_i\}_{i,P}$ where $i \in \mathbb{N}$ and P varies over Seifert fibered pieces of M and \tilde{P} varies over all components of $f^{-1}(P)$. Notice that any two members of the covering $\{\tilde{P}_i\}_{i,P}$ either do not intersect or intersect along some tori boundary or along some incompressible annuli on tori boundary components. Hence union of any finitely many members of this covering is a compact irreducible (possibly not connected) 3-manifold with incompressible tori boundary components.

Now we can choose $0 < r_1 < r_2 \cdots < r_m < \cdots$ and $i_1^l, i_2^l, \dots, i_{k_l}^l$, $l = 1, 2, \dots$, so that $r_m \rightarrow \infty$ and

$$\cup_{j=1}^{j=k_l-1} \tilde{P}_{i_j^{l-1}} \subset \delta^{-1}([-r_l, r_l]) \subset \cup_{j=1}^{j=k_l} \tilde{P}_{i_j^l}$$

and $\cup_{j=1}^{j=k_l} \tilde{P}_{i_j^l}$ is connected. Write $N_l = \cup_{j=1}^{j=k_l} \tilde{P}_{i_j^l}$.

This completes the proof of the Lemma. \square

Proof of Theorem 4.8. Let M be a closed 3-manifold which satisfies Condition* virtually. That is there is a regular finite sheeted covering \tilde{M} of M which satisfies Condition*. Let L be the group of covering transformation of $\tilde{M} \rightarrow M$. We have $\pi_1(M) < \pi_1(\tilde{M}) \wr L$. Thus we only need to check the FIC for $\pi_1(\tilde{M}) \wr L$. There are two cases to consider.

Case A. $[\pi_1(\tilde{M}), \pi_1(\tilde{M})]$ is finitely generated. Since $H_1(\tilde{M}, \mathbb{Z})$ is infinite we can apply Proposition 8.1 to deduce that $[\pi_1(\tilde{M}), \pi_1(\tilde{M})]$ is isomorphic to the fundamental group of a compact surface (say F). If $[\pi_1(\tilde{M}), \pi_1(\tilde{M})]$ is infinite cyclic then $\pi_1(M)$ is virtually poly- \mathbb{Z} and hence the FICwF is true for $\pi_1(M)$ by Proposition 2.1. Otherwise we apply Proposition 8.1 again to conclude that $H_1(\tilde{M}, \mathbb{Z})$ is virtually infinite cyclic. Let C be an infinite cyclic normal subgroup of $H_1(\tilde{M}, \mathbb{Z})$ of finite index and let $A : \pi_1(\tilde{M}) \rightarrow H_1(\tilde{M}, \mathbb{Z})$ denotes the abelianization homomorphism. Hence $A^{-1}(C)$ is a finite index subgroup of $\pi_1(\tilde{M})$ and thus isomorphic to the fundamental group of a closed 3-manifold. Also since $A^{-1}(C)$ has a finitely generated normal subgroup with infinite cyclic quotient, by Stallings' fibration theorem $A^{-1}(C)$ is isomorphic to the fundamental group of a closed 3-manifolds fibering over the circle.

Let $C' = H_1(\tilde{M}, \mathbb{Z})$. Then $A^{-1}(C') < A^{-1}(C) \wr (C'/C)$. Hence by Lemma 5.4 $\pi_1(\tilde{M}) \wr L = (A^{-1}(C')) \wr L < (A^{-1}(C) \wr (C'/C)) \wr L < (A^{-1}(C))^{((C'/C) \times L)} \wr ((C'/C) \wr L)$. By Theorem 4.5 (item 4) and Lemma 5.1 the FIC is true for $(A^{-1}(C))^{((C'/C) \times L)} \wr ((C'/C) \wr L)$ and hence for $\pi_1(\tilde{M}) \wr L$ also. This completes the proof of the theorem in this case.

Case B. M satisfies (b) of Condition*. By Remark 4.2 $[\pi_1(\tilde{M}), \pi_1(\tilde{M})]$ is infinitely generated.

By Lemma 8.1 it follows that $A^{-1}(Z) < \lim_i G_i^Z$ where each G_i^Z is isomorphic to the fundamental group of a compact irreducible 3-manifold with incompressible tori boundary components and Z is any infinite (or trivial) cyclic subgroup of $H_1(\tilde{M}, \mathbb{Z})$ and A denotes the homomorphism $\pi_1(\tilde{M}) \rightarrow H_1(\tilde{M}, \mathbb{Z})$. By Corollary 4.1 the FIC is true for $G_i^Z \wr L$ for any finite group L .

Now note that we are in the setting of the Lemma 6.6.

This completes the proof of Theorem 4.8. \square

Proof of Theorem 4.9. Let us first check that the hypothesis of the theorem implies that N is irreducible where N is a finite sheeted covering of M realizing G . That is $G = \pi_1(N)$. If N is not irreducible, then either G is a nontrivial free product or N is an \mathbb{S}^2 -bundle over the circle. Since H is nontrivial and of infinite index in G the second case does not occur. Since H is a nontrivial finitely generated normal subgroup of the free product G , by [[16], lemma 11.2] H should have finite index in G . Again this is a contradiction. Hence N is irreducible.

The main ingredient behind the proof is [[16], theorem 11.1]. We need the following proposition which is an application of [[16], theorems 11.1, 11.6] and the proof of some special cases of the Seifert fiber space conjecture.

Proposition 8.1. *Let N be a compact 3-manifold so that there is an exact sequence $1 \rightarrow H \rightarrow \pi_1(N) \rightarrow Q \rightarrow 1$, where H is finitely generated and Q is infinite. Then the followings hold.*

- H is the fundamental group of a compact surface.
- if $H \simeq \mathbb{Z}$ then either N is Seifert fibered or there is a finite sheeted Seifert fibered covering of N .
- if H is not infinite cyclic then Q is virtually cyclic.
- if Q is cyclic then either $\mathcal{P}(N)$ fibers over the circle with fiber a compact surface with fundamental group isomorphic to H or $\mathcal{P}(N)$ is homotopy equivalent to $\mathbb{P}^2 \times \mathbb{S}^1$.

Remark 8.1. For definition of the Poincaré associate $\mathcal{P}(N)$ of N see [[16], p. 88]. For our purpose it is enough to recall that N and $\mathcal{P}(N)$ have the same fundamental group.

Proof of Proposition 8.1. The first and third conclusions are (1) and (2) of [[16], theorem 11.1] respectively. For the second one note that N is a compact irreducible 3-manifold with an infinite cyclic normal subgroup H . If H is central and N is orientable then by a by now well-known theorem N is Seifert fibered. Otherwise by [[16], remark 11.2] there is a two sheeted covering $p : \tilde{N} \rightarrow N$ so that $H \cap p_*(\pi_1(\tilde{N}))$ is central in $p_*(\pi_1(\tilde{N}))$. Hence using the previous case we conclude the proof of the second item. The fourth item is (1) of [[16], theorem 11.6]. This completes the proof of the Proposition. \square

To prove the Theorem there are now two cases to consider.

Case A. H is infinite cyclic. Since H is now an infinite cyclic normal subgroup of the fundamental group of the compact irreducible 3-manifold N , by Proposition 8.1 either N is Seifert fibered or there

is a finite sheeted cover of N which is Seifert fibered. Hence there is a finite sheeted covering of M which is Seifert fibered. Therefore the FICwF is true for $\pi_1(M)$ by Theorem 4.6.

Case B. H is not infinite cyclic. In this case by the third case of Proposition 8.1, Q is virtually cyclic. Let Q' be an infinite cyclic normal subgroup of Q of finite index. Let $p : G \rightarrow Q$ be the projection. Then applying the fourth case of Proposition 8.1 to the exact sequence $1 \rightarrow H \rightarrow p^{-1}(Q') \rightarrow Q' \rightarrow 1$ we get that $\mathcal{P}(M')$ either fibers over the circle or $\mathcal{P}(M')$ is homotopically equivalent to $\mathbb{P}^2 \times \mathbb{S}^1$. Here M' is the finite sheeted covering of N corresponding to the subgroup $p^{-1}(Q')$. Hence we get that either $\mathcal{P}(M)$ has a finite sheeted covering which fibers over the circle or $\pi_1(M)$ is virtually infinite cyclic. Using Theorem 4.5 we complete the proof of the theorem. \square

9. PROPERTIES OF THE CLASS \mathcal{C}

This section is devoted to describe some properties of 3-manifolds belonging to the class \mathcal{C} . Let \mathcal{C}' be the subclass of \mathcal{C} consisting of 3-manifolds which do not contain any properly embedded essential annulus. We establish that, in some suitable sense (defined below), \mathcal{C}' is a basis of \mathcal{C} , that is \mathcal{C} can be obtained from \mathcal{C}' by some fundamental operations.

Let M be a compact connected 3-manifold with nonempty boundary. Let A_1 and A_2 be two disjoint incompressible annuli on ∂M . Let us denote by $M \#_A$ the manifold obtained by identifying A_1 with A_2 by some diffeomorphism of A_1 and A_2 . Now let M_1 and M_2 be two compact connected 3-manifolds with nonempty boundary. Let $A_1 \subset \partial M_1$ and $A_2 \subset \partial M_2$ be two incompressible annuli. We denote by $M_1 \#_A M_2$ the manifold obtained by identifying A_1 and A_2 by some diffeomorphism. On the fundamental group level the first case corresponds to HNN -extension and the second one corresponds to generalized free product case.

Definition 9.1. We call $M \#_A$ an *HNN -connected sum* along an annulus and $M_1 \#_A M_2$ a *generalized connected sum* along an annulus.

Proposition 9.1. *Let $M \in \mathcal{C}$. Then there are $M_1, \dots, M_k \in \mathcal{C}'$ and M is obtained from M_1, \dots, M_k by successive application of HNN and generalized connected sum along annuli on the boundary components of M_i .*

Proof. If M has no properly embedded essential annulus then there is nothing to prove. So assume there is a properly embedded essential annulus $A \subset M$. Let $N = \overline{M - A}$. Then N is an irreducible 3-manifold

and it is easy to show by Euler characteristic calculation that all the boundary components of N are surfaces of genus ≥ 2 . Incompressibility of the boundary also follows easily. Hence $N \in \mathcal{C}$. If N is connected then M is obtained from N by HNN -connected sum along an annulus and if N is disconnected then M is obtained from N by generalized connected sum along an annulus. Now again we apply the same procedure if N contains a properly embedded essential annulus. Applying standard (reducing complexity) tricks from 3-manifold topology it follows now that after a finite stage we get manifolds which contain no properly embedded essential annuli. This proves the proposition. \square

Remark 9.1. Note that the splitting theorem in [[19], p. 157] states that any compact Haken 3-manifold contains a two-sided incompressible 2-manifold which is unique up to ambient isotopy and the components of this 2-manifolds are either tori or annuli and none of them is boundary parallel. Also after cutting the manifold along this 2-manifold one gets pieces which are either Seifert fibered or simple. Here we would like to remark that the proof of the Proposition 9.1 consists of only cutting along these unique class of annuli.

In the following we give examples of members of \mathcal{C} which contains essential annuli and the boundary is not totally geodesic.

Example 9.1. Let S be a closed orientable surface of genus ≥ 2 . Let $M = S \times [0, 1]$. Then $M \in \mathcal{C}$. It is also easy to show that the interior of M supports a complete hyperbolic metric. We check the following two properties about M .

- M contains properly embedded essential annulus.
- at least one of the boundary components of M is not totally geodesic.

The first property is obvious. For example take any simple closed curve $\gamma \subset S$ and then consider $\gamma \times [0, 1]$.

For the second property, suppose on the contrary, that the boundary components are totally geodesic. Identify $S \times \{0\}$ with $S \times \{1\}$ by the identity map and let N be the resulting manifold. Since $S \times \{0\}$ and $S \times \{1\}$ are totally geodesic in M , N also supports a hyperbolic metric. This is a contradiction because by Preissman's theorem ([3], theorem 3.2) any nontrivial abelian subgroup of $\pi_1(N)$ is infinite cyclic, on the other hand $\pi_1(N)$ contains a free abelian subgroup on 2 generators.

10. ON A -REGULAR RIEMANNIAN MANIFOLDS AND 3-MANIFOLDS

This section is devoted to give some examples of noncompact Riemannian manifolds with A -regular Riemannian metric. The basic material on Riemannian geometry we use here are given in any standard book (for example see [17] or [3]). We follow the notations and conventions of [3].

Proposition 10.1. *Let M be a A -regular Riemannian manifold. Then the product metric on the n -fold product M^n is also A -regular.*

Proof. It is enough to prove the Proposition for $n = 2$. Let $\chi(K)$ be the space of all vector fields on a manifold K and let $R_K(X, Y, Z, W) = \langle R_K(X, Y)Z, W \rangle$, $X, Y, Z, W \in \chi(K)$ denote the curvature tensor if K is a Riemannian manifold. Here $\langle \cdot, \cdot \rangle$ denotes the Riemannian metric on K and $R_K(X, Y)Z$ is the curvature. Let $p_1 : M \times M \rightarrow M$ and $p_2 : M \times M \rightarrow M$ be the first and the second projection. For any vector field V on $M \times M$ let $V_1 = (p_1)_*(V)$ and $V_2 = (p_2)_*(V)$ where $(p_i)_* : T(M \times M) \rightarrow TM$ is the induced map on the tangent bundle for each $i = 1, 2$. Let $X, Y, Z, W \in \chi(M \times M)$ and $X = X_1 + X_2, Y = Y_1 + Y_2, Z = Z_1 + Z_2, W = W_1 + W_2$ be the orthogonal decomposition of the vector fields as defined above. where $X_i, Y_i, Z_i, W_i \in \chi(M)$ for $i = 1, 2$.

Now we need the following easily verified lemma.

Lemma 10.1. *Let M and N be two Riemannian manifolds and $p_1 : M \times N \rightarrow M$ and $p_2 : M \times N \rightarrow N$ be the two projections. Let X, Y, Z and W be vector fields on $M \times N$ such that $(p_2)_*(X) = 0 = (p_1)_*(Y)$ and $(p_2)_*(W) = 0 = (p_1)_*(Z)$ and let ∇ denotes the Riemannian connection on $M \times N$. Then the followings are true.*

- $\nabla_X Y = \nabla_W Z = [X, Y] = [W, Z] = 0$.
- let $f : N \rightarrow \mathbb{R}$ be a smooth function then $X(f \circ p_2) = 0$.

Now recall the definition of the curvature. For a Riemannian manifold K ,

$$R_K(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X, Y]} Z.$$

Expanding the curvature linearly and using the above Lemma we deduce that

$$R_{M \times M}(X, Y, Z, W) = R_{M \times M}(X_1, Y_1, Z_1, W_1) + R_{M \times M}(X_2, Y_2, Z_2, W_2).$$

Fix X, Y, Z and W . Let $(p, q) \in M \times M$. Then the two real numbers

$$R_{M \times M}(X_1(p, q), Y_1(p, q), Z_1(p, q), W_1(p, q))$$

and

$$R_M(X_1(p), Y_1(p), Z_1(p), W_1(p))$$

are equal and similarly

$$R_{M \times M}(X_2(p, q), Y_2(p, q), Z_2(p, q), W_2(p, q))$$

and

$$R_M(X_2(q), Y_2(q), Z_2(q), W_2(q))$$

are equal.

Hence

$$\begin{aligned} & R_{M \times M}(X^1(p, q), \dots, X^4(p, q)) \\ &= R_M(X_1^1(p), \dots, X_1^4(p)) + R_M(X_2^1(q), \dots, X_2^4(q)). \end{aligned}$$

Now using the above Lemma and by linearity one easily checks the following equality.

$$\begin{aligned} & \nabla^i R_{M \times M}(X^1(p, q), X^2(p, q), \dots, X^{i+4}(p, q)) \\ &= \nabla^i R_M(X_1^1(p), \dots, X_1^{i+4}(p)) + \nabla^i R_M(X_2^1(q), \dots, X_2^{i+4}(q)) \end{aligned}$$

for each i . Here for each $j = 1, 2, \dots, i+4$, X_1^j and X_2^j denotes the two orthogonal components of X^j , that is $X^j = X_1^j + X_2^j$.

Thus if M is an A -regular Riemannian manifold and A_i are nonnegative constants such that $|\nabla^i R_M| \leq A_i$ for each i then $|\nabla^i R_{M \times M}| \leq 2A_i$ for each i . Hence $M \times M$ is also A -regular.

This proves the Proposition. \square

In the proof of the above proposition, in fact we have shown something more.

Corollary 10.1. *Let M and N be two complete Riemannian manifolds with A -regular metric. Then the product metric on $M \times N$ is also A -regular.*

Proposition 10.2. *Let M be a complete Riemannian manifold and outside a compact subset it has constant sectional curvature. Then the metric is A -regular.*

Proof. Note that the curvature tensor is completely determined by the sectional curvatures ([3], chapter 4, lemma 3.3). It is now easy to check that the covariant derivatives of the curvature tensor is zero outside a compact subset. And obviously they are bounded on a compact subset. This proves the Proposition. \square

An application of Proposition 10.2 is Proposition 4.1.

Proof of Proposition 4.1. Note that M is a Haken 3-manifold and hence it admits a JSJT decomposition (along tori) into Seifert fibered and hyperbolic pieces. Since boundary components of Seifert fibered pieces are tori the pieces abutting the boundary components of M are all hyperbolic. This proves the first assertion. On the other hand by [23]

the hyperbolic metric in the hyperbolic pieces can be deformed near the tori boundary components and a Riemannian metric of nonpositive sectional curvature can be given in the interior of M so that outside a compact subset of M the metric has constant -1 sectional curvature. Hence this Riemannian metric on M is A -regular by Proposition 10.2. Also if we consider a finite product $M \times \cdots \times M$ then the product Riemannian metric is also A -regular and nonpositively curved by Proposition 10.1. Now since $\pi_1(M) \wr G \simeq \pi_1(M^G) \rtimes G$ it follows that $\pi_1(M) \wr G$ is an A -group. \square

11. EXAMPLES OF 3-MANIFOLDS SATISFYING CONDITION*

In this section we work out some examples of graph manifolds for which Condition* is satisfied. The simplest example of this sort is given in the following way.

Example 11.1. Let F_1 and F_2 be two compact orientable non-simply connected surfaces and $P_i = F_i \times \mathbb{S}^1$ for $i = 1, 2$. Assume F_i has one boundary component for $i = 1, 2$. Let for each $i = 1, 2$, (λ_i, μ_i) denotes the basis of $\pi_1(\partial P_i) \simeq \mathbb{Z} \times \mathbb{Z}$ where μ_i represents ∂F_i and λ_i represents the second factor \mathbb{S}^1 . Since ∂F_i has one component, μ_i represents an element of the commutator subgroup of $\pi_1(F_i)$ for $i = 1, 2$.

Let V be the closed manifold obtained by identifying P_1 and P_2 along their boundary tori by a diffeomorphism f . Then V is a graph manifold.

The diffeomorphism f induces the following isomorphism

$$f_* : \pi_1(\partial P_1) \simeq \mathbb{Z} \times \mathbb{Z} \rightarrow \pi_1(\partial P_2) \simeq \mathbb{Z} \times \mathbb{Z}$$

and hence there are pairs of integers (p_1, q_1) and (p_2, q_2) with the following properties.

- $p_1 q_2 - p_2 q_1 = 1$
- f_* sends λ_1 to $\lambda_2^{p_1} \mu_2^{q_1}$ and μ_1 to $\lambda_2^{p_2} \mu_2^{q_2}$.

Proposition 11.1. *Under the above notations the followings are true.*

- if $p_2 \neq 0$ then V satisfies (b) in the definition of Condition*.
- if $p_2 = 0 = q_1$ then V is a product.

Proof. By an easy calculation we see that in the fundamental group of V the following are satisfied.

- $\lambda_1^{p_2} = \mu_1^{p_1} \mu_2^{-1}$.
- $\lambda_2^{p_2} = \mu_1 \mu_2^{-q_2}$.

If $q_1 = 0 = p_2$ then it easily follows that V is a product of a compact surface and the circle.

Now if $p_2 \neq 0$ then we have that $\lambda_2^{p_2} = \mu_1 \mu_2^{-q_2}$ and $\lambda_1^{p_2} = \mu_1^{p_1} \mu_2^{-1}$ in $\pi_1(V)$. Since μ_1 and μ_2 both represents elements of $[\pi_1(V), \pi_1(V)]$ we see that $[\pi_1(V), \pi_1(V)] \cap \pi_1(P_2)$ contains $\lambda_2^{p_2}$ and hence $[\pi_1(V), \pi_1(V)] \cap \pi_1(P_2)$ is not free. This follows from the fact that $[\pi_1(V), \pi_1(V)] \cap [\pi_1(F_2), \pi_1(F_2)]$ is nontrivial. Similarly $[\pi_1(V), \pi_1(V)] \cap \pi_1(P_1)$ is also not free.

This proves the Proposition. \square

Similar ideas as in the above Proposition can be used to construct examples satisfying (b) of Condition*.

Remark 11.1. Here we remark that there are only finitely many ways to identify the torus boundary of P_1 with that of P_2 to produce a closed graph manifold admitting a metric of nonpositive sectional curvature (see [21]).

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